

# Conductivity between Luttinger liquids: coupled chains and bilayer graphene

Vieri Mastropietro<sup>1</sup>

<sup>1</sup>*Università di Roma Tor Vergata, Viale della Ricerca Scientifica 00133 Roma, Italy, EU*

The conductivity properties between Luttinger liquids are analyzed by exact Renormalization Group methods. We prove that in a two chain system or in a model of bilayer graphene, described by two coupled fermionic honeycomb lattices interacting with a gauge field, the transverse optical conductivity at finite temperature is anomalous and decreasing together with the frequency as a power law with Luttinger liquid exponent.

PACS numbers: 05.10.Cc, 05.30.Fk, 71.10.Fd

## INTRODUCTION

The many body interaction in fermionic systems can destroy the electron-like nature of the elementary excitations, a fact which can have deep consequences on the transport properties. This was pointed out first by Anderson [1], who got evidence that the coherent transport between Luttinger liquids due to particle hopping is strongly depressed with respect to the non interacting case. One of the first applications of this idea was, see [2, 3], an explanation of the c-axis anomalous conductivity between planes in high  $T_c$  superconductors, with the *assumption* of Luttinger liquid behavior in the planes described by 2D square lattice Hubbard models; while intriguing, this theory suffers the fact that no convincing theoretical evidence has been found up to now to substantiate such assumption. Subsequently, the attention was focused to one dimensional systems, where Luttinger liquid behavior is surely present; in addition to a theoretical interest, the main physical motivation was that coupled fermionic chains well describe quasi-one dimensional organic conductors, see *e.g.* [4]. At zero temperature Renormalization Group or bosonization analysis apparently indicate that the hopping can destroy Luttinger liquid behavior in several regimes, see *e.g.* [5–17]. On the other hand at higher temperatures the system still shows Luttinger liquid properties [18–21]. The transverse conductivity between weakly coupled fermionic chains at temperatures and frequencies greater than the hopping was shown to be  $\sigma^\perp(\omega) \sim \omega^\alpha$  with  $\alpha$  related to the Luttinger liquid exponent  $\eta$ ; in particular the value of the exponent  $\alpha$  was claimed to be  $\alpha = 2\eta$ , in [22], where a tunnelling approach was followed, and in [23], by using the Kubo formula; on the other hand, in [24] the *different* value  $\alpha = 2\eta - 1$  was found, by using dynamical mean field theory. The reason of these discrepancies relies on the fact that the computations were done in the so called Tomonaga-Luttinger or *g-ology* approximation, in which the fermions close to the two Fermi points are described in terms of massless Dirac particles. This simplifies the computations and allows the use of powerful techniques like bosonization, but introduces spurious ultraviolet divergences in the conductivity which therefore

needs a regularization, and different regularizations produces different results (see *e.g.* the discussion in [24] after eq.(3)).

In conclusion, there are at present no firm results on the conductivity between higher dimensional Luttinger liquids, and even in one dimension there are still ambiguities due to approximations and regularizations. In this paper the conductivity properties between Luttinger liquids at finite temperature are analyzed by the *exact* Renormalization Group methods developed starting from [25]; indeed the suggestion of using such techniques for this problem dates back to Anderson himself [1] but their full development required a long time. In the case of coupled spinless chains we get an exact expression for the transverse conductivity (the lattice furnishes the natural cut-off), which is, at temperatures and frequencies greater than the hopping

$$\sigma_\beta^\perp(\omega_n) \sim t^2 \omega_n^{2\eta-1} \quad (1)$$

if  $\omega_n = \frac{2\pi}{\beta}(n + \frac{1}{2})$ . The analysis is based on the implementation of Ward Identities in the Renormalization Group, with rigorous bounds for the corrections due to the lattice [26]. Note also that the computation of the parallel conductivity gives in this regime  $\sigma_\beta^\parallel(\omega_n) \sim \omega_n^{-1}$ , that is no anomalous exponent appears in the frequency dependence in that case.

In addition to justifying the anomalous exponent predicted by mean field theory in [24] for the two chain problem, the exact Renormalization Group methods can provide for the first time evidence for anomalous transverse conductivity between *bidimensional* Luttinger liquids, for which bosonization cannot be applied. Electrons on the honeycomb lattice interacting with an  $U(1)$  gauge field, representing the retarded e.m. interaction or the effect of disorder or ripples [27], have Luttinger liquid behavior. Indeed this system was first analyzed in the continuum Dirac approximation in the early work [28], where evidence of Luttinger liquid was found based on second order perturbation theory. Later on, in [29, 30] Luttinger liquid behavior was established *at any order* and taking rigorously into account the honeycomb lattice, by implementing lattice Ward Identities in the Renormalization Group scheme. The behavior of the two point function

is similar to the one of the spinless chain; the wave function renormalization has a power law with exponent  $\eta$ . By coupling two interacting fermionic honeycomb lattices by an hopping term we get a model for *bilayer graphene* [31]. The zero temperature properties of such system are rather complex and still not completely understood, see *e.g.* [32–34] and the review [35]. However, as in the case of coupled chains, at finite temperature and frequencies the Luttinger liquid behavior of the uncoupled system can reveal itself by the transverse conductivity; in particular we will show that for temperatures and frequencies greater than the hopping

$$\sigma_{\beta}^{\perp}(\omega_n) \sim t^2 \omega_n^{2\eta} \quad (2)$$

while is essentially constant in the non interacting case. This confirms for the first time in two dimensions the Anderson idea: coherent transport between Luttinger liquids is depressed with respect to non interacting systems or Fermi liquids. Moreover, the presence of an anomalous Luttinger liquid exponents in the frequency dependence of the transverse conductivity of bilayer graphene could be revealed in future experiments.

The paper is organized in the following way. In §II we derive the transverse conductivity for the two chain model; in §III we derive the transverse conductivity for bilayer graphene. In App A the computation of the conductivity in the non interacting case is presented.

## THE TWO CHAIN MODEL

We consider a two chain model described as two one dimensional interacting spinless fermionic systems coupled by an hopping term; the Hamiltonian is

$$H = H_1 + H_2 + P \quad (3)$$

where, for  $i = 1, 2$ ,  $H_i = H_i^{(0)} + V_i$

$$\begin{aligned} H_i^{(0)} &= -\frac{1}{2} \sum_{x=1}^{L-1} (a_{x+1,i}^+ a_{x,i}^- + a_{x,i}^+ a_{x+1,i}^-) \\ V_i &= -\lambda \sum_{x,y=1}^{L-1} v(x-y) a_{x,i}^+ a_{x,i}^- a_{y,i}^+ a_{y,i}^- \end{aligned} \quad (4)$$

and

$$P = -t \sum_{x=1}^{L-1} [a_{x,1}^+ a_{x,2}^- + a_{x,2}^+ a_{x,1}^-] \quad (5)$$

where  $a_{x,i}^{\pm}$  are fermionic operators and  $|v(x)| \leq e^{-\kappa|x|}$ . Either  $\lambda$  and  $t$  are assumed to be *small*.

As usual we can introduce the interaction with an e.m. field with a Peierls substitution  $H \rightarrow H(A)$  with  $V_i(A) = V$  and, if  $A = (A^{\parallel}, A^{\perp})$ ,

$$H_i^{(0)}(A) = -\sum_x \frac{1}{2} (a_{x+1,i}^+ e^{iA_x^{\parallel}} a_{x,i}^- + a_{x,i}^+ e^{-iA_x^{\parallel}} a_{x+1,i}^-) \quad (6)$$

and

$$P(A) = -t \sum_x (a_{x,1}^+ e^{iA_x^{\perp}} a_{x,2}^- + a_{x,2}^+ e^{-iA_x^{\perp}} a_{x,1}^-) \quad (7)$$

The *parallel current* is defined as

$$\frac{\partial H(A)}{\partial A_x^{\parallel}} = j_x^{P,\parallel} + A_x^{\parallel} j_x^{D,\parallel} + O((A^{\parallel})^2) \quad (8)$$

where  $j_{x,i}^{P,\parallel}$  and  $j_{x,i}^{D,\parallel}$  are called respectively *paramagnetic* and *diamagnetic* part of the current and are given by

$$\begin{aligned} j_x^{P,\parallel} &= \sum_{i=1}^2 \frac{1}{2i} (a_{x+1,i}^+ a_{x,i}^- - a_{x,i}^+ a_{x+1,i}^-) \\ j_x^{D,\parallel} &= \sum_{i=1}^2 \frac{1}{2} (a_{x+1,i}^+ a_{x,i}^- + a_{x,i}^+ a_{x+1,i}^-) \end{aligned} \quad (9)$$

The *transverse current* is defined as

$$\frac{\partial H(A)}{\partial A_x^{\perp}} = j_x^{P,\perp} + A_x^{\perp} j_{x,i}^{D,\perp} + O((A^{\perp})^2) \quad (10)$$

where  $j_x^{P,\perp}$  and  $j_x^{D,\perp}$  are called respectively *paramagnetic* and *diamagnetic* part of the current and are given by

$$\begin{aligned} j_x^{P,\perp} &= \frac{t}{i} (a_{x,1}^+ a_{x,2}^- - a_{x,2}^+ a_{x,1}^-) \\ j_x^{D,\perp} &= t (a_{x,1}^+ a_{x,2}^- + a_{x,2}^+ a_{x,1}^-) \end{aligned} \quad (11)$$

Finally the fermionic density is  $\rho_x = a_{1,x}^+ a_{1,x} + a_{2,x}^+ a_{2,x}$ .

If  $\mathbf{p} = (\omega_n, p)$ ,  $\omega_n = \frac{2\pi}{\beta} n$ ,  $p = \frac{2\pi}{L} m$ , the transverse conductivity at finite temperature is given by

$$\sigma_{\beta}^{\perp}(\omega_n) = \frac{1}{\omega_n} \lim_{p \rightarrow 0} [-\langle \hat{j}_{\mathbf{p}}^{P,\perp}; \hat{j}_{-\mathbf{p}}^{P,\perp} \rangle + \langle j_x^{D,\perp} \rangle] \quad (12)$$

where, if  $A = O_{\mathbf{x}_1} \cdots O_{\mathbf{x}_n}$ ,  $\langle A \rangle = \frac{\text{Tr}[e^{-\beta H} \mathbf{T}(A)]}{\text{Tr}[e^{-\beta H}]}$ , with  $\mathbf{T}$  being the time order product and  $T$  denotes truncation. An analogous definition holds for the parallel conductivity.

In order to compute the conductivity (12) it is convenient to introduce a Grassmann integral representation for the correlation; we introduce the following *generating functional*

$$e^{\mathcal{W}_{t,\lambda}(A^{\perp})} = \int P(d\psi) e^{-\mathcal{V}(\psi) - B(A^{\perp}, \psi)} \quad (13)$$

where  $\psi_{\mathbf{x},i}^{\pm}$  are Grassmann variables,  $\mathbf{x} = (x_0, x)$ ,  $P(d\psi)$  is the Grassmann integration with propagator  $\delta_{i,j} \hat{g}(\mathbf{k})$

$$\hat{g}(\mathbf{k}) = \frac{1}{-ik_0 + \cos k - \cos p_F} \quad (14)$$

with  $k_0 = \frac{2\pi}{\beta}(n + \frac{1}{2})$ ,  $\mathcal{V}$  is the interaction given by, if  $\int d\mathbf{x} = \int_{-\beta/2}^{\beta/2} dx_0 \sum_x$

$$\begin{aligned} \mathcal{V}(\psi) &= -\lambda \sum_{i=1}^2 \int d\mathbf{x} dy v(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x},i}^+ \psi_{\mathbf{x},i} \psi_{\mathbf{y},i}^+ \psi_{\mathbf{y},i} \\ &\quad -\nu \sum_{i=1}^2 \int d\mathbf{x} \psi_{\mathbf{x},i}^+ \psi_{\mathbf{x},i} - t \int d\mathbf{x} (\psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- + \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},1}^-) \end{aligned} \quad (15)$$

with  $v(\mathbf{x}-\mathbf{y}) = \delta(x_0 - y_0)v(x-y)$  and  $\nu$  is a counterterm which is introduced to take into account the renormalization of the chemical potential; moreover

$$B(\psi, A^\perp) = -i \int d\mathbf{x} A_\mathbf{x}^\perp [\psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- - \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},1}^-]$$

Defining  $H_t(\mathbf{x})$  the Fourier transform of  $\langle \hat{j}_\mathbf{p}^\perp; \hat{j}_\mathbf{p}^\perp \rangle$  we can write

$$H_t(\mathbf{x}) = t^2 \frac{\partial^2}{\partial A_\mathbf{x}^\perp \partial A_0^\perp} \mathcal{W}_{t,\lambda}(A^\perp)|_0 \quad (16)$$

The analysis of the functional integral (13) will be done by Renormalization Group (RG), integrating smaller and smaller momentum scales. The hopping  $t$  introduces an intrinsic scale in the RG analysis. For scales greater than the (renormalized) hopping, it is natural to treat the hopping as a perturbation using the chain variables  $\psi^\pm$ ; on the other hand at smaller scales the hopping cannot be considered a perturbation and it is convenient to use the variables

$$\hat{b}_{\mathbf{k},1} = \frac{1}{\sqrt{2}}[\hat{\psi}_{\mathbf{k},1} + \hat{\psi}_{\mathbf{k},2}] \quad \hat{b}_{\mathbf{k},2} = \frac{1}{\sqrt{2}}[\hat{\psi}_{\mathbf{k},1} - \hat{\psi}_{\mathbf{k},2}] \quad (17)$$

in terms of which the free action is diagonal but the  $b_{\mathbf{k},1}^\pm, b_{\mathbf{k},2}^\pm$  have different Fermi momentum. Note that the temperature acts an infrared cut-off so that for temperatures not too small only the first regime is present.

The first step of the RG analysis is the decomposition of the propagator  $g(\mathbf{k})$  as a sum of propagators supported close to the two Fermi points  $\pm p_F$  and more and more singular in the infrared region, labelled by a quasi particle index  $\alpha = \pm$  (labelling the Fermi points) and by an integer  $h \leq 0$ :

$$\hat{g}(\mathbf{k}) = \hat{g}^{(1)}(\mathbf{k}) + \sum_{h=h_\beta}^0 \sum_{\alpha=\pm} \hat{g}_\alpha^{(h)}(\mathbf{k} - \mathbf{p}_F^\alpha) \quad (18)$$

with  $\mathbf{p}_F^\alpha = (0, p_F)$ ,  $g_\alpha^{(h)}$  supported on  $2^{h-1} \leq |\mathbf{k} - \mathbf{p}_F^\alpha| \leq 2^{h+1}$  and  $g^{(1)}(\mathbf{k})$  has support far from the Fermi points. Note that  $2^{h_\beta} \sim \pi/\beta$ ; the fact that the temperature is finite implies that there is a finite number of scales.

The RG integration procedure is defined recursively in the following way. Assume that we have integrated the fields  $\psi_i^{(1)}, \psi_{i,\alpha}^{(0)}, \dots, \psi_{i,\alpha}^{(h+1)}$ ; we get

$$e^{\mathcal{W}_{t,\lambda}(A^\perp)} = e^{F_h(A^\perp)} \quad (19)$$

$$\int P(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - B^{(h)}(A^\perp, \sqrt{Z_h}\psi^{(\leq h)})}$$

where  $P(d\psi^{(\leq h)})$  is the Grassmanian quadratic integration with propagator given by

$$g_\alpha^{(\leq h)}(\mathbf{x}) = \frac{1}{\beta L} \frac{1}{Z_h} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_h(\mathbf{k})}{-ik_0 + \cos p_F - \cos(k + \alpha p_F)} \quad (20)$$

with  $\chi_h(\mathbf{k})$  is a smooth cut-off function with support  $|\mathbf{k} - \alpha \mathbf{p}_F| \leq 2^{h+1}$  and  $Z_h$  is the wave function renormalization. The single scale propagator  $g_\alpha^{(h)}(\mathbf{x})$  is obtained from  $\hat{g}_\alpha^{(\leq h)}(\mathbf{k})$  replacing  $\chi_h(\mathbf{k})$  with  $f_h(\mathbf{k})$  with support  $2^{h-1} \leq |\mathbf{k}'| \leq 2^{h+1}$ ,  $\mathbf{k} = (k_0, k')$ ,  $k = k' + \alpha p_F$ ,  $k'$  is the momentum measured from the Fermi point. It can be written as

$$g_\alpha^{(h)}(\mathbf{x}) = e^{i\alpha p_F x} \frac{1}{\beta L} \frac{1}{Z_h} \sum_{\mathbf{k}} e^{i\mathbf{k}'\mathbf{x}} \frac{f_h(\mathbf{k}')}{-ik_0 + \alpha v_F k'} + r^{(h)}(\mathbf{x}) \quad (21)$$

with  $r^{(h)}(\mathbf{x})$  with the same decay properties as  $g_\alpha^{(h)}(\mathbf{x})$  with an extra factor  $2^h$ ; therefore, the more we are close to the Fermi momenta (*i.e.*  $-h$  is large), the more  $r^{(h)}(\mathbf{x})$  is a small correction and the propagator is essentially coinciding with the one of a massless Dirac particle. Finally  $\mathcal{V}^{(h)}$  is the *effective potential* expressed by a sum of monomials of order  $n$  in the fields  $\psi^{(\leq h)}$  multiplied times a kernel  $W_{n,0}^{(h)}$ , while  $B^{(h)}$  is sum of monomials of order  $n$  in  $\psi$  and  $m$  in  $A^{(\perp)}$  with kernels  $W_{n,m}^{(h)}$ . According to power counting, using that  $\hat{g}_\alpha(\mathbf{k})^{(h)} \sim 2^{-h}$  and  $\int d\mathbf{k} \hat{g}_\alpha^{(h)}(\mathbf{k}) \sim 2^h$ , the "naive" scaling dimension of such monomials is

$$D = 2 - n/2 - m \quad (22)$$

In the RG analysis we have to decompose  $\mathcal{V}^{(h)}$  (a similar decomposition must be done also for  $B^{(h)}$ ) as

$$\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)} \quad (23)$$

with  $\mathcal{R} = 1 - \mathcal{L}$ ;  $\mathcal{L}\mathcal{V}^{(h)}$  is the *relevant* or *marginal* part of the effective interaction while  $\mathcal{R}\mathcal{V}^{(h)}$  is the *irrelevant* part. Generally this decomposition is dictated by the naive scaling dimension (22);  $\mathcal{L}$  should select the terms with positive or vanishing dimension  $D$ . However, if the temperature verifies the condition

$$2^{h_\beta} > t_{h_\beta} \quad (24)$$

where  $t_h$  is the hopping at scale  $h$ , there is an improvement with respect to naive power counting, and certain terms which are dimensionally relevant or marginal are indeed irrelevant. In order to verify this fact, we can split the kernels as  $W_{n,m}^{(h)} = W_{n,m}^{(a)(h)} + W_{n,m}^{(b)(h)}$  where  $W_{n,m}^{(a)(h)}$  is obtained from  $W_{n,m}^{(h)}$  setting  $t = 0$ . In the case  $n = 4, m = 0$  (with vanishing scaling dimension)

$$\mathcal{L}\hat{W}_{4,0}^{(h)}(\underline{\mathbf{k}}') = \hat{W}_{4,0}^{(a)(h)}(\underline{\mathbf{0}}) \quad (25)$$

so that

$$\mathcal{R}\hat{W}_{4,0}^{(h)}(\underline{\mathbf{k}}') = [\hat{W}_{4,0}^{(a)(h)}(\mathbf{k}') - \hat{W}_{4,0}^{(a)(h)}(\underline{\mathbf{0}})] + \hat{W}_{4,0}^{(b)(h)}(\mathbf{k}') \quad (26)$$

The first term in the r.h.s. of (26) can be rewritten as  $\underline{\mathbf{k}}' \cdot \underline{\partial} W_{n,m}^{(a)(h)}$ , and this produces an improvement  $\sim 2^{h'-h}$

in the bound of the kernel, if  $h'$  is the scale of the momentum, which is sufficient to make it irrelevant. Similarly the second term in (26), namely  $\hat{W}_{4,0}^{(b)(h)}(\mathbf{k}')$ , has an extra  $t_h 2^{-h} \leq 2^{h_\beta - h}$  with respect to the bound for  $W_{4,0}^{(h)}$ , which again is enough to make it irrelevant; therefore, the true marginal contribution is given by the r.h.s. of (25). Therefore the only marginal quartic terms involve fermions with the same chain index, and that the corresponding effective coupling coincide with the one of the uncoupled  $t = 0$  case.

Similarly we define, for the terms with  $n = 2$  and the same chain index

$$\mathcal{L}\hat{W}_{2,0}^{(h)}(\mathbf{k}') = \hat{W}_{2,0}^{(a)(h)}(\mathbf{0}) + \mathbf{k}' \partial \hat{W}_{2,0}^{(a)(h)}(\mathbf{0}) \quad (27)$$

Note that  $W_{2,0}^{(b)(h)}$  has an extra  $(t_h 2^{-h})^2$  (there are no terms linear in  $t_h$  by conservation of the chain index). Finally, if  $n = 2, m = 0$  and the fermionic fields have different chain index

$$\mathcal{L}\hat{W}_{2,0}^{(h)}(\mathbf{k}') = \hat{W}_{2,0}^{(h)}(\mathbf{0}) \quad (28)$$

Note that the terms with  $n = 2$  and an extra derivative are irrelevant as they have at least an extra  $t_h 2^{-h}$ . Therefore

$$\begin{aligned} \mathcal{L}\mathcal{V}^{(h)}(\psi) &= \sum_{i=1}^2 [\lambda_h \int d\mathbf{x} [\psi_{\mathbf{x},i,+}^+ \psi_{\mathbf{x},i,+}^- + \psi_{\mathbf{x},i,-}^+ \psi_{\mathbf{x},i,-}^-] \\ &+ 2^h \sum_{\alpha=\pm} \nu_h \psi_{\mathbf{x},i,\alpha}^+ \psi_{\mathbf{x},i,\alpha}^- + \delta_h \sum_{\alpha=\pm} \psi_{\mathbf{x},i,\alpha}^+ \psi_{\mathbf{x},i,\alpha}^-] + \\ &+ t_h \sum_{\alpha=\pm} \int d\mathbf{x} [\psi_{\mathbf{x},1,\alpha}^+ \psi_{\mathbf{x},2,\alpha}^- + \psi_{\mathbf{x},2,\alpha}^+ \psi_{\mathbf{x},1,\alpha}^-] \end{aligned} \quad (29)$$

In the above expression  $\lambda_h$  represents the effective interaction at momentum scale  $h$ ,  $\delta_h$  the effective Fermi velocity,  $\nu_h$  the shift of the chemical potential and  $t_h$  the effective hopping. By definition,  $Z_h, \lambda_h, \nu_h, \delta_h$  are *exactly the same* as in the theory with  $t = 0$ . It is possible to choose  $\nu$  so that  $\nu_h$  remain small for any  $h$ . By combining Ward-Identities at each Renormalization group iteration together with Schwinger-Dyson equation it follows, see [26], that

$$\lambda_h \rightarrow_{h \rightarrow -\infty} \lambda_{-\infty}(\lambda) \quad \delta_h \rightarrow_{h \rightarrow -\infty} \delta_{-\infty}(\lambda) \quad (30)$$

with  $\lambda_{-\infty}(\lambda), \delta_{-\infty}(\lambda)$  analytic functions of  $\lambda$ ; moreover

$$Z_h \sim 2^{-\eta h} \quad (31)$$

with  $\eta$  analytic in  $\lambda$  and  $\eta = a\lambda^2 + O(\lambda^3)$  with  $a > 0$ . Moreover, in [26] (and references therein) it is also proven that kernels  $W_{n,m}^{(h)}$  are *analytic functions* of  $\{\lambda_k, \nu_k, \delta_k, t_k\}_{k \geq h}$ : analyticity (implying the "non-perturbative" nature of the method) is a very non trivial property obtained exploiting anticommutativity properties of Grassmann variables, via *Gram inequality* for

determinants and Bridges-Battle-Federbush formula for truncated expectations.

Regarding the flow of  $t_h$  we obtain

$$t_{h-1} = \frac{Z_h}{Z_{h-1}}(t_h + \beta_t^{(h)}) \quad (32)$$

with  $|\beta_t^{(h)}| \leq C_1 t_h \lambda^2 (t_h 2^{-h})^2$ . It is easy to see by induction that  $|Z_h t_h - t| \leq C_2 t |\lambda|$ . Assume indeed that it is true for  $k \geq h$ ; therefore for  $\lambda, t$  small enough

$$|t_{h-1} Z_{h-1} - t| \leq 2t C_1 \lambda^2 \sum_{k=h}^0 (t_k 2^{-k})^2 \quad (33)$$

from which, using (24), the inductive assumption follows. Note that the effective hopping, even if *relevant* in the RG sense according to naive power counting, remains small in this region of temperatures. Moreover, from (24) we obtain the condition between the temperature and the hopping

$$\beta^{-1} \geq t^{\frac{1}{1-\eta}} (1 + O(\lambda^2)) \quad (34)$$

Regarding the effective source  $B^{(h)}$ , we define

$$\mathcal{L}W_{2,1}^{(h)}(\mathbf{k}', \mathbf{p}) = W_{2,1}^{(a)(h)}(\mathbf{0}, \mathbf{0}) = 1 \quad (35)$$

Indeed the graphs contributing to  $W_{2,1}^{(a)(h)}(\mathbf{0})$  are one particle reducible (as the interaction involves only fermions from the same chain) and  $g^{(h)}(\mathbf{k}')|_{\mathbf{k}'=0} = 0$ . Therefore (assuming that  $A_{\mathbf{p}}^\perp$  has small support around  $\mathbf{p} = 0$ )

$$\begin{aligned} \mathcal{L}B^{(h)}(A^\perp, \sqrt{Z_h}\psi) &= \\ -i \sum_{i=1}^2 \sum_{\alpha=\pm} \int d\mathbf{x} A_{\mathbf{x}}^\perp (\psi_{\mathbf{x},1,\alpha}^+ \psi_{\mathbf{x},2,\alpha}^- + \psi_{\mathbf{x},2,\alpha}^+ \psi_{\mathbf{x},1,\alpha}^-) \end{aligned} \quad (36)$$

As the flow of the effective parameters corresponding to the relevant and marginal operators is bounded, the following bound is obtained, for  $\beta$  verifying (34)

$$\frac{1}{L\beta} \int d\mathbf{x} |W_{n,m}^{(h)}(\mathbf{x})| \leq C 2^{h(2-\frac{n}{2}-m)} \quad (37)$$

In order to compute the conductivity we have to separate the terms proportional to  $t^2$  in both the paramagnetic and diamagnetic contributions to (12) from the rest. We write the current-current correlation as

$$t^{-2} H_t(\mathbf{x}) = \frac{\partial^2 \mathcal{W}_{0,\lambda}}{\partial A_{\mathbf{x}}^\perp \partial A_{\mathbf{0}}^\perp} |_0 + t^{-2} \tilde{H}_t(\mathbf{x}) \quad (38)$$

where the first term in the r.h.s. is independent from  $t$  and, from (37)

$$\int d\mathbf{x} |\tilde{H}_t(\mathbf{x})| \leq C t^2 \sum_{h=h_\beta}^0 \left(\frac{t}{2^h}\right)^2 Z_h^{-4} \leq 2t^2 C (t\beta^{1-2\eta})^2 \quad (39)$$

In order to compute the conductivity we still have to compute  $\langle j_x^{D,\perp} \rangle$ ; introducing the generating functional

$$e^{\bar{W}_{t,\lambda}(J)} = \int P(d\psi) e^{-\mathcal{V}(\psi) - t \int d\mathbf{x} J_{\mathbf{x}} h_{\mathbf{x}}} \quad (40)$$

where  $h_{\mathbf{x}} = \psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- + \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},1}^-$  we get

$$\langle j_x^{D,\perp} \rangle = \frac{\partial \bar{W}_{t,\lambda}}{\partial J_x} \Big|_0 = t^2 \int d\mathbf{x}_1 \frac{\partial^2 \bar{W}_{0,\lambda}}{\partial J_{\mathbf{x}} \partial J_{\mathbf{x}_1}} \Big|_0 + \Delta \quad (41)$$

where

$$\Delta = \sum_{n=3}^{\infty} \frac{t^{n+1}}{n!} \int d\mathbf{x}_1 \dots \int d\mathbf{x}_n \frac{\partial^{n+1} \bar{W}_{0,\lambda}}{\partial J_{\mathbf{x}} \partial J_{\mathbf{x}_1} \dots \partial J_{\mathbf{x}_n}} \Big|_0 \quad (42)$$

and only  $n$  odd contribute. From the analogue of (37) the l.h.s. is bounded by the sum over  $h$  of  $\sum_{n=3}^{\infty} t^{n+1} 2^{-h(n-1)} Z_h^{-4}$  so that, for  $t\beta$  small

$$|\Delta| \leq C_1 t^2 \sum_{h=h_\beta}^0 (2^{-h} t Z_h^{-2})^2 \leq C_2 t^2 (t\beta^{1-2\eta})^2 \quad (43)$$

Note finally that

$$- \frac{\partial^2 \bar{W}_{0,\lambda}}{\partial A_{\mathbf{x}}^+ \partial A_{\mathbf{0}}^+} \Big|_0 + \int d\mathbf{x}_1 \frac{\partial^2 \bar{W}_{0,\lambda}}{\partial J_{\mathbf{x}} \partial J_{\mathbf{x}_1}} \Big|_0 \quad (44)$$

can be rewritten as

$$\langle \psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- - \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},2}^-; \psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- - \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},2}^- \rangle_{0,\lambda} + \int d\mathbf{y} \langle \psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- + \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},2}^-; \psi_{\mathbf{y},1}^+ \psi_{\mathbf{y},2}^- + \psi_{\mathbf{x},2}^+ \psi_{\mathbf{y},2}^- \rangle_{0,\lambda}$$

or equivalently

$$\langle \psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- - \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},2}^-; \psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- - \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},2}^- \rangle_{0,\lambda} - \int d\mathbf{y} \langle \psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},2}^- - \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},2}^-; \psi_{\mathbf{y},1}^+ \psi_{\mathbf{y},2}^- - \psi_{\mathbf{x},2}^+ \psi_{\mathbf{y},2}^- \rangle_{0,\lambda}$$

This means that there is an important cancellation between the paramagnetic and diamagnetic part of the conductivity; indeed

$$- \langle j_{\mathbf{p}}^{P,\perp}; j_{-\mathbf{p}}^{P,\perp} \rangle + \langle j_x^{D,\perp} \rangle = t^2 \int d\mathbf{x} (e^{i\omega_n x_0} - 1) \langle j_{\mathbf{x},D}^{\perp} j_{\mathbf{y},D}^{\perp} \rangle_{0,\lambda} + O(t^2 (t\beta^{1-2\eta})^2) \quad (45)$$

where

$$\langle j_{\mathbf{x},D}^{\perp}; j_{\mathbf{y},D}^{\perp} \rangle_{0,\lambda} = \langle \psi_{\mathbf{x},1}^+ \psi_{\mathbf{y},1}^+ \rangle_{0,\lambda} \langle \psi_{\mathbf{y},2}^+ \psi_{\mathbf{x},2}^- \rangle_{0,\lambda}$$

Therefore for  $\omega_n$  small

$$\begin{aligned} & | \int d\mathbf{x} (e^{i\omega_n x_0} - 1) \langle j_{\mathbf{x},D}^{\perp} j_{\mathbf{y},D}^{\perp} \rangle_{0,\lambda} | \\ & \leq C_1 \int_{|\mathbf{x}| \leq |\omega_n|^{-1}} d\mathbf{x} \frac{|x_0 \omega_n|}{1 + |\mathbf{x}|^{2+2\eta}} + \\ & C_1 \int_{|\mathbf{x}| \geq |\omega_n|^{-1}} d\mathbf{x} \frac{1}{1 + |\mathbf{x}|^{2+2\eta}} \leq \frac{C_2}{\eta} |\omega_n|^{2\eta} \end{aligned} \quad (46)$$

In conclusion the transverse conductivity is given by (1), for  $t \ll \beta^{-1} \ll \omega_n \ll 1$ . In the non interacting case  $t^{-2} \omega_n \sigma_{\beta}^{\perp}(\omega_n) \sim \frac{2}{\pi \sin p_F}$  so that we can conclude that the presence of the inter-chain interaction decreases the transverse conductivity in this regime.

The transverse conductivity should be compared with the parallel conductivity  $\sigma_{\beta}^{\parallel}$ , defined as in (12) with  $j^{D,\parallel}, j^{P,\parallel}$  replacing  $j^{D,\perp}, j^{P,\perp}$ . For  $t \ll \beta^{-1} \ll \omega_n \ll 1$  one gets

$$\omega_n \sigma_{\beta}^{\parallel}(\omega_n) \sim 2 \frac{v_F K}{\pi} \quad (47)$$

where  $v_F = \sin p_F + O(\lambda)$  is the interacting Fermi velocity and  $K$  is the Luttinger liquid parameter  $2\eta = K + K^{-1} - 2$ . The above formula, which is part of the Haldane's Luttinger liquid conjecture [36], can be derived by a Renormalization Group analysis analogue to the one described above, see [37]. The parallel current-current correlation is obtained by a generating functional similar to (13) in which  $B(A^{\perp}, \psi)$  is replaced by  $B(A^{\parallel}, \psi) = \int d\mathbf{x} A_{\mathbf{x}}^{\parallel} j_{\mathbf{x}}^{P,\parallel}$ ; after the integration of the fields  $\psi_i^{(1)}, \psi_{i,\alpha}^{(0)}, \dots, \psi_{i,\alpha}^{(h+1)}$  we get an expression similar to (20) with  $B^{(h)}(A^{\perp}, \sqrt{Z_h} \psi)$  replaced by  $B^{(h)}(A^{\parallel}, \sqrt{Z_h} \psi)$  with

$$\mathcal{L} B^{(h)}(A^{\parallel}, \sqrt{Z_h} \psi) = Z_h^{(1)} \int d\mathbf{x} j_{\mathbf{x}}^{P,\parallel}(\leq h) \quad (48)$$

and

$$\frac{Z^{(1)}}{Z_h} = 1 + O(\lambda) \quad (49)$$

as a consequence of a Ward identity. Therefore the renormalization of the parallel current is proportional to the wave function renormalization (while there is no renormalization of the transverse current, see (35)) and this explains why anomalous power law exponents do not appear in the frequency dependence of the parallel conductivity. In conclusion, from (1) and (47) we see that  $\eta$  can be independently determined in experiments on two chain systems either from the amplitude of the parallel conductivity or from the exponent in the orthogonal conductivity.

As a final remark, we stress that the above analysis is true only for temperatures greater than  $\sim t^{\frac{1}{1-\eta}}$ ; at lowest temperature the RG analysis would be identical to the previous one up to scale  $t^{\frac{1}{1-\eta}}$ , but at lowest scales one should perform the change of variables (17); the system would be described in terms of two fermions with different Fermi momenta (the difference is  $O(t^{\frac{1}{1-\eta}})$ ). In this second regime the power counting improvement described above is not valid, and this produces several (not a single one, as in spinless Luttinger liquids) effective quartic couplings with a generically unbounded RG flow.

## BILAYER GRAPHENE

An analysis similar to the previous one can be repeated for a model of bilayer graphene, described in terms of electrons on the honeycomb lattice interacting through an  $U(1)$  quantized gauge field, which can represent either the e.m. interaction or the effects of ripples or disorder, see e.g. [27].

We introduce creation and annihilation fermionic operators  $\psi_{\vec{x},i}^{\pm} = (a_{\vec{x},i}^{\pm}, b_{\vec{x}+\vec{\delta}_1,i}^{\pm}) = |\mathcal{B}|^{-1} \int_{\vec{k} \in \mathcal{B}} d\vec{k} \psi_{\vec{k},i,\sigma}^{\pm} e^{\pm i\vec{k}\vec{x}}$  for electrons with plane index  $i = 1, 2$  and sitting at the sites of the two triangular sublattices  $\Lambda_A$  and  $\Lambda_B$  of a honeycomb lattice; we assume that  $\Lambda_A$  has basis vectors  $\vec{l}_{1,2} = \frac{1}{2}(3, \pm\sqrt{3})$  and that  $\Lambda_B = \Lambda_A + \vec{\delta}_j$ , with  $\vec{\delta}_1 = (1, 0)$  and  $\vec{\delta}_{2,3} = \frac{1}{2}(-1, \pm\sqrt{3})$  the nearest neighbor vectors;  $\mathcal{B}$  is the first Brillouin zone and  $|\mathcal{B}| = \frac{8\pi^2}{3\sqrt{3}}$ . In the absence of e.m. interaction, the Hamiltonian is

$$H = H_1 + H_2 + P \quad (50)$$

where

$$H_i = - \sum_{\substack{\vec{x} \in \Lambda_A \\ j=1,2,3}} a_{\vec{x},i}^{+} b_{\vec{x}+\vec{\delta}_j,i}^{-} + c.c. \quad (51)$$

describes the hopping of fermions in the plane and

$$P = -t \sum_{\vec{x} \in \Lambda_A} [a_{\vec{x},1}^{+} a_{\vec{x},2}^{-} + a_{\vec{x},2}^{+} a_{\vec{x},1}^{-} + b_{\vec{x}+\vec{\delta}_1,1}^{+} b_{\vec{x}+\vec{\delta}_1,2}^{-} + b_{\vec{x}+\vec{\delta}_1,2}^{+} b_{\vec{x}+\vec{\delta}_1,1}^{-}] \quad (52)$$

describes the fermionic hopping from one plane to another; either  $e$  and  $t$  will be assumed small. The interaction with a transverse classical e.m. field is introduced via the Peierls substitution. If  $A^{\perp}$  is a classical external field

$$P = -t \sum_{\vec{x} \in \Lambda_A} [a_{\vec{x},1}^{+} e^{iA_{\vec{x}}^{\perp}} a_{\vec{x},2}^{-} + a_{\vec{x},2}^{+} e^{-iA_{\vec{x}}^{\perp}} a_{\vec{x},1}^{-} + b_{\vec{x}+\vec{\delta}_1,1}^{+} e^{iA_{\vec{x}+\vec{\delta}_1}^{\perp}} b_{\vec{x}+\vec{\delta}_1,2}^{-} + b_{\vec{x}+\vec{\delta}_1,2}^{+} e^{-iA_{\vec{x}+\vec{\delta}_1}^{\perp}} b_{\vec{x}+\vec{\delta}_1,1}^{-}] \quad (53)$$

the paramagnetic and diamagnetic part of the transverse current are

$$\begin{aligned} j_{\vec{x}}^{P,\perp} &= \frac{\partial H(A)}{\partial A_{\vec{x}}^{\perp}} \Big|_0 = \frac{t}{i} [a_{\vec{x},1}^{+} a_{\vec{x},2}^{-} + b_{\vec{x}+\vec{\delta}_1,1}^{+} b_{\vec{x}+\vec{\delta}_1,2}^{-} - a_{\vec{x},2}^{+} a_{\vec{x},1}^{-} - b_{\vec{x}+\vec{\delta}_1,2}^{+} b_{\vec{x}+\vec{\delta}_1,1}^{-}] \\ j_{\vec{x}}^{D,\perp} &= \frac{\partial^2 H(A)}{\partial^2 A_{\vec{x}}^{\parallel}} = t [a_{\vec{x},1}^{+} a_{\vec{x},2}^{-} + b_{\vec{x}+\vec{\delta}_1,1}^{+} b_{\vec{x}+\vec{\delta}_1,2}^{-} + a_{\vec{x},2}^{+} a_{\vec{x},1}^{-} + b_{\vec{x}+\vec{\delta}_1,2}^{+} b_{\vec{x}+\vec{\delta}_1,1}^{-}] \end{aligned} \quad (54)$$

and the transverse conductivity is defined as in (12) divided by  $\frac{3\sqrt{3}}{2}$ , the area of the hexagonal cell of the honeycomb lattice.

We assume now that the electrons interact through an  $U(1)$  gauge field; the current-current correlation is obtained from the following generating functional

$$e^{\mathcal{W}_{t,e}(A^{\perp})} = \int P(d\psi) P(dA) e^{-\mathcal{V}(\psi,A) - B(A^{\perp},\psi)} \quad (55)$$

where  $\psi = (a, b)$  a couple of Grassmann variables (with slight abuse of notation, the Grassmann and the fermionic operators are denoted with the same symbol),  $P(d\psi)$  is the fermionic gaussian integration for  $\psi_{\vec{k},i}^{\pm}$  ( $i = 1, 2$  denotes the plane),  $\vec{k} = k_0, \vec{k}$ ,  $k_0 = \frac{2\pi}{\beta}(n + \frac{1}{2})$ , with propagator  $\delta_{i,j}g(\mathbf{k})$  with

$$g^{-1}(\mathbf{k}) = - \begin{pmatrix} ik_0 & v_0 \Omega^*(\vec{k}) \\ v_0 \Omega(\vec{k}) & ik_0 \end{pmatrix}, \quad (56)$$

and  $v_0 = \frac{3}{2}$  and  $\Omega(\vec{k}) = \frac{2}{3} \sum_{j=1}^3 e^{i\vec{k}(\vec{\delta}_j - \vec{\delta}_1)}$ . The complex dispersion relation  $\Omega(\vec{k})$  vanishes only at the two *Fermi points*  $\vec{p}_F^{\pm} = (\frac{2\pi}{3}, \pm \frac{2\pi}{3\sqrt{3}})$  and close to them assumes the form of a relativistic dispersion relation  $\Omega(\vec{p}_F^{\pm} + \vec{k}') \simeq ik'_1 \pm k'_2$ . Moreover

$$\begin{aligned} \mathcal{V} &= - \int d\mathbf{x} [a_{\vec{x},i,\sigma}^{+} b_{\vec{x}+\vec{\delta}_j,\sigma} e^{ie \int_0^1 \vec{\delta}_j \cdot \vec{A}_i(\mathbf{x} + s\vec{\delta}_j) ds} + c.c.] + \\ &\int d\mathbf{x} A_{\vec{x},i}^{(0)} a_{\vec{x},i}^{+} a_{\vec{x},i,\sigma} + A_{\vec{x}+\vec{\delta}_1,i}^{(0)} b_{\vec{x}+\vec{\delta}_1,i}^{+} b_{\vec{x}+\vec{\delta}_1,i} \end{aligned} \quad (57)$$

where  $\int d\mathbf{x} \equiv \sum_{\vec{x} \in \Lambda_A} \int dx_0$  and  $A_{\mu,i} = (\vec{A}_i, A_i^{(0)})$  is a boson field with propagator  $\delta_{i,j}w(\mathbf{p})$  with

$$w(\mathbf{p}) = \frac{\chi(\sqrt{\omega_n^2 + c^2 p^2})}{\sqrt{\omega_n^2 + c^2 p^2}} \quad (58)$$

where  $\chi$  is a cut-off function forbidding momenta either too large and smaller than the temperature. Finally the source term is given by

$$B(A^{\perp}, \psi) = \int d\mathbf{x} A_{\mathbf{x}}^{\perp} j_{\mathbf{x}}^{P,\perp} \quad (59)$$

We proceed exactly as in the previous case, writing the photon propagator as sum of propagator more and more singular in the infrared region, and the fermionic propagator as a sum of propagators supported close to the two Fermi points  $\vec{p}_F^{\pm} = (\frac{2\pi}{3}, \pm \frac{2\pi}{3\sqrt{3}})$ , labelled by a quasi particle index  $\alpha = \pm$  (labelling the Fermi points) and by an integer  $h \leq 0$ :

$$\begin{aligned} w(\mathbf{p}) &= \sum_{h=h_{\beta}}^1 w^{(h)}(\mathbf{p}) \\ g(\mathbf{k}) &= g^{(1)}(\mathbf{k}) + \sum_{h=h_{\beta}}^0 \sum_{\alpha=\pm} g_{\alpha}^{(h)}(\mathbf{k} - \mathbf{p}_F^{\alpha}) \end{aligned} \quad (60)$$

with  $w^{(h)}(\mathbf{p})$  supported in  $2^{h-1} \leq |\mathbf{p}| \leq 2^{h+1}$ ,  $g_{\alpha}^{(h)}$  supported on  $2^{h-1} \leq |\mathbf{k} - \mathbf{p}_F^{\alpha}| \leq 2^{h+1}$  and  $g^{(1)}(\mathbf{k})$  has support far from the Fermi points.

Assume that we have integrated out the fields  $\psi^{(1)}, \dots, A^{(h+1)}, \psi^{(h+1)}$ ,  $h \geq h_\beta$  so that

$$e^{W(A^\perp)} = e^{F_h(A^\perp)} \quad (61)$$

$$\int P(d\psi^{(\leq h)}) \int P(dA^{(\leq h)}) e^{\mathcal{V}^{(h)}(A, \sqrt{Z_h}\psi) + B_h(A^\perp, \sqrt{Z_h}\psi)}$$

where  $P(dA^{(\leq h)})$  is the gauge field integration with propagator  $\delta_{i,j}\delta_{\mu,\nu}w^{\leq h}(\mathbf{p})$ , with  $w^{\leq h}(\mathbf{p}) = \sum_{k=-\infty}^h w^{(k)}(\mathbf{p})$ , while  $P(d\psi^{(\leq h)})$  is the integration of the fermionic field  $\psi_{i,\alpha}$  with propagator  $\delta_{i,i'}\delta_{\alpha,\alpha'}g_{\alpha}^{\leq h}(\mathbf{k} - \mathbf{p}_F^{(\alpha)})$  with, if  $\mathbf{k}' = \mathbf{k} - \mathbf{p}_F^{(\alpha)}$ ,  $g_{\alpha}^{(\leq h)}(\mathbf{k}') =$

$$= \frac{\chi_h(\mathbf{k}')}{Z_h} \begin{pmatrix} -ik_0 & v_h(ik'_1 - \alpha k'_2) \\ v_h(-ik'_1 - \alpha k'_2) & -ik_0 \end{pmatrix}^{-1} (1 + R_\omega^{(h)}). \quad (62)$$

In Eq.(62)  $\chi_h(\mathbf{k}')$  is a cut-off function with support in  $|\mathbf{k}'| \leq 2^h$  and  $|R_\omega^{(h)}(\mathbf{k}')| \leq C|\mathbf{k}'|^\theta$  for some  $\theta > 0$ , while  $Z_h$  and  $v_h$  are, respectively, the effective wave function renormalization and Fermi velocity on scale  $h$ .

The effective potential  $\mathcal{V}^{(h)} + B^{(h)}$  expressed by a sum of monomials of order  $n$  in the fields  $\psi^{(\leq h)}$ ,  $m$  in  $A^{(\mu)(\leq h)}$  and  $l$  in  $A^\perp$ , multiplied by kernels  $W_{n,m,l}^{(h)}$ . According to power counting the naive scaling dimension of such monomials is

$$D = 3 - n - m - l \quad (63)$$

Again there is a dimensional improvement with respect to power counting if we are in a range of temperatures larger than the hopping, that is  $2^{h_\beta} > t_{h_\beta}$  where  $t_h$  is the hopping at scale  $h$ . We can split the kernels as  $W_{n,m,l}^{(a)(h)} = W_{n,m,l}^{(a)(h)} + W_{n,m,l}^{(b)(h)}$  where  $W_{n,m,l}^{(b)(h)}$  is obtained from  $W_{n,m,l}^{(a)(h)}$  setting  $t = 0$ . We define the  $\mathcal{L}$  operator in the following way

$$\mathcal{L}\hat{W}_{2,1,0}^{(h)}(\mathbf{k}') = \hat{W}_{2,1,0}^{(a)(h)}(0) \quad (64)$$

Note indeed that the extra  $t_h 2^{-h} \leq 2^{h_\beta - h}$  in  $W_{2,1,0}^{(b)(h)}(0)$  is sufficient to make it irrelevant. Regarding the terms quadratic in the gauge fields,  $\mathcal{L}\hat{W}_{0,2,0}^{(h)}(\mathbf{p}) = \hat{W}_{0,2,0}^{(a)(h)}(0) + \mathbf{p}\partial\hat{W}_{0,2,0}^{(a)(h)}(0)$ , where we have used that  $\hat{W}_{0,2,0}^{(b)(h)}(0)$  has an extra  $(2^{-h}t_h)^2$  with respect to the naive dimension; moreover either  $\hat{W}_{0,2,0}^{(a)(h)}(0)$  and  $\partial\hat{W}_{0,2,0}^{(a)(h)}(0)$  are vanishing as consequence of the gauge symmetry, see [30]. Finally the terms quadratic in the fermionic variables, if they have the same plane index then  $\mathcal{L}\hat{W}_{2,0,0}^{(h)}(\mathbf{k}') = \hat{W}_{2,0,0}^{(a)(h)}(0) + \mathbf{k}'\partial\hat{W}_{2,0,0}^{(a)(h)}(\mathbf{k}')$  where we have used that in  $\hat{W}_{2,0,0}^{(b)(h)}$  there is an extra gain  $O((t_h 2^{-h})^2)$ , due to the conservation of the plane index  $i$ . On the other hand for the quadratic terms with different plane index

$$\mathcal{L}\hat{W}_{2,0,0}^{(h)}(\mathbf{k}') = \hat{W}_{2,0,0}^{(h)}(\mathbf{0}) \quad (65)$$

Therefore

$$\mathcal{L}\mathcal{V}^{(h)}(A, \psi) = t_h \int d\mathbf{x} j_{\mathbf{x}}^{D,\perp} + \sum_{\mu,i,\alpha} \bar{e}_{\mu,h} \int \frac{d\mathbf{k}}{(2\pi)|B|} \frac{d\mathbf{p}}{(2\pi)^3} \psi_{\mathbf{k}+\mathbf{p},i,\alpha}^+ \Gamma_\mu^\omega \psi_{\mathbf{k},i,\alpha}^- A_i^\mu(\mathbf{p}) \quad (66)$$

where:  $\bar{e}_{0,h} = e_{0,h}$ ,  $\bar{e}_{i,h} = v_h e_{i,h}$ ,  $e_{1,h} = e_{2,h}$  (thanks to discrete rotational symmetry),  $\Gamma_\mu^\omega := \Gamma_\mu(\vec{p}_F^\omega, \vec{0})$  (with  $\Gamma_0^\omega = -iI$ ,  $\Gamma_1^\omega = -\sigma_2$ ,  $\Gamma_2^\omega = -\omega\sigma_1$  and  $\sigma_{1,2}$  the first two Pauli matrices) and  $\mathcal{R}\mathcal{V}^{(h)}$  a sum of terms that are irrelevant in the RG sense. By construction the flow of the effective parameters is the same as in the model with  $t = 0$ ; it was show in [30], by a rigorous implementation of Ward Identities in the RG scheme, that the effective charges flows to a line of fixed points and the Fermi velocity increases up to the light velocity

$$e_h \rightarrow e_{-\infty} \quad v_h \rightarrow c \quad (67)$$

Moreover, the wave function renormalization  $Z_h$  diverges with anomalous exponents

$$Z_h \sim 2^{-\eta h} \quad \eta = \frac{e^2}{12\pi^2} + \dots \quad (68)$$

Finally regarding the flow of  $t_h$  we obtain

$$t_{h-1} = \frac{Z_h}{Z_{h-1}}(t_h + \beta_t^{(h)}) \quad (69)$$

with  $|\beta_t^{(h)}| \leq C_1 e^6 t_h [\frac{t_h}{2h}]^2$ , and again by induction  $|Z_h t_h - t| \leq C_2 t e^6$ . We assume that the temperature verifies (24) which implies  $\beta^{-1} \geq t^{\frac{1}{1-\eta}}(1 + O(e^2))$ . Regarding the effective source  $B^{(h)}$ , we define  $\mathcal{L}W_{2,0,1}^{(h)}(\mathbf{k}', \mathbf{p}) = W_{2,0,1}^{(a)(h)}(\mathbf{0}, \mathbf{0}) = 1$  as again the graphs contributing to  $W_{2,0,1}^{(a)(h)}(\mathbf{0})$  are one particle reducible and  $g^{(k)}(\mathbf{k}')|_{\mathbf{k}'=0} = 0$ . As the flow of the effective parameters corresponding to the relevant and marginal operators is bounded, the following bound is obtained, for  $h \geq h_\beta$  (order by order in the renormalized expansion)

$$\frac{1}{\Lambda\beta} \int d\mathbf{x} |W_{n,m,l}^{(h)}(\mathbf{x})| \leq C 2^{h(3-n-m-l)} \quad (70)$$

Using the same notation as in (38)

$$\int d\mathbf{x} |x_0| |\tilde{H}_t(\mathbf{x})| \leq C t^2 \sum_{h=h_\beta}^0 \left(\frac{t}{2h}\right)^2 Z_h^{-4} \leq 2t^2 C (t\beta^{1-2\eta})^2 \quad (71)$$

Moreover, as in the previous case we introduce a generating functional  $\bar{W}_{t,e}(J)$  with source  $t \int d\mathbf{x} J_{\mathbf{x}} h_{\mathbf{x}}$  where  $j_{\mathbf{x}}^{D,\perp} = t h_{\mathbf{x}}$  we get

$$\langle j_x^{D,\perp} \rangle = t^2 \int d\mathbf{x}_1 \frac{\partial^2 \bar{W}_{0,e}}{\partial J_{\mathbf{x}} \partial J_{\mathbf{x}_1}}|_0 + \Delta \quad (72)$$

From the analogue of (70) the l.h.s. is bounded by the sum over  $h$  of  $\sum_{n=3}^{\infty} t^{n+1} 2^{-h(n-2)} Z_h^{-4}$  so that, for  $t\beta$  small

$$\omega_n^{-1} |\Delta| \leq t^2 \sum_{h=h_\beta}^0 \beta 2^h \left[ \frac{t 2^{-h}}{Z_h^2} \right]^2 \leq C t^2 (t\beta^{1-2\eta})^2 \quad (73)$$

Note finally that

$$\langle j_{\mathbf{x}}^{P,\perp}; j_{\mathbf{y}}^{P,\perp} \rangle_{0,e} = \langle j_{\mathbf{x}}^{D,\perp}; j_{\mathbf{y}}^{D,\perp} \rangle_{0,e} \quad (74)$$

and

$$\begin{aligned} & \left| \int d\mathbf{x} x_0 (e^{i\omega_n x_0} - 1) \langle j_{\mathbf{x},D}^{\perp}; j_{\mathbf{y},D}^{\perp} \rangle_{0,\lambda} \right| \\ & \leq C_1 \int_{|\mathbf{x}| \leq \omega_n^{-1}} d\mathbf{x} |x_0| \frac{|x_0 \omega_n|}{1 + |\mathbf{x}|^{4+2\eta}} + \\ & C_1 \int_{|\mathbf{x}| \geq \omega_n^{-1}} d^3 \mathbf{x} |x_0| \frac{1}{1 + |\mathbf{x}|^{4+2\eta}} \leq \frac{C_2}{\eta} |\omega_n|^{2\eta} \end{aligned}$$

Therefore the conductivity in the interacting case is given by (2) for  $t \ll \beta^{-1} \ll \omega_n \ll 1$ , that is the transverse conductivity decreases with the frequency with the anomalous exponent  $2\eta$ . In absence of planar interaction  $t^{-2} \sigma_{\beta}^{\perp}(\omega_n) \sim \frac{1}{2}$ , so that we can conclude that the presence of planar long range interaction producing Luttinger liquid behavior decreases the transverse conductivity. Note also that the parallel conductivity does not display any anomalous power law, as a consequence of a Ward Identity implying the analogue of (49), see [30].

## APPENDIX: THE NON INTERACTING CASE

In the case of the two chain model if  $\lambda = 0$

$$\begin{aligned} t^{-2} \omega_n \sigma_{\beta}^{\perp}(\omega_n) &= \left[ \int d\mathbf{k} g(\mathbf{k} + \mathbf{p}) g(\mathbf{k}) - \right. \\ & \left. \int d\mathbf{k} g(\mathbf{k}) g(\mathbf{k}) \right]_{p=0} + O((\beta t)^2) \end{aligned}$$

Note that

$$\lim_{\beta, L \rightarrow \infty} \frac{1}{L\beta} \sum_{\mathbf{k}} g(\mathbf{k} + \mathbf{p}) g(\mathbf{k})|_{p=0} = 0 \quad (75)$$

while

$$\lim_{\beta, L \rightarrow \infty} \frac{1}{L\beta} \sum_{\mathbf{k}} g(\mathbf{k}) g(\mathbf{k}) = \frac{2}{\pi \sin p_F} \quad (76)$$

In the case of bilayer graphene, we get  $t^{-2} \sigma_{\beta}^{\perp}(\omega_n) =$

$$\frac{1}{\omega_n} \int \frac{dk_0}{(2\pi)} d\vec{k} [F(\mathbf{k}, \mathbf{k} + \mathbf{p}) - F(\mathbf{k}, \mathbf{k})]_{\vec{p}=0} + O((t\beta)^2)$$

where

$$\begin{aligned} F(\mathbf{k}_1, \mathbf{k}_2) &= 2[g_{11}(\mathbf{k}_1)g_{11}(\mathbf{k}_2) + g_{22}(\mathbf{k}_1)g_{22}(\mathbf{k}_2) \\ &+ g_{12}(\mathbf{k}_1)g_{12}(\mathbf{k}_2) + g_{21}(\mathbf{k}_1)g_{21}(\mathbf{k}_2)] \end{aligned} \quad (77)$$

The first term in the r.h.s. can be written as its value for  $\beta = \infty$  plus a rest  $O(\beta^{-1})$ ; the integral in the limit  $\beta = \infty$  can be decomposed in a part integrated in the region  $|\Omega(\vec{k})| \leq \varepsilon$  and  $|\Omega(\vec{k})| \geq \varepsilon$ ; the second term is vanishing for  $\omega_n = 0$  while in the first the contribution from the first two terms in (77) are vanishing by parity, while the rest gives  $\frac{1}{2}$  at vanishing external frequency.

*Acknowledgements* The Author gratefully acknowledges financial support from the ERC Starting Grant CoMBoS-239694

- 
- [1] P.W.Anderson *Phys. Rev. Lett.* 67, 27, 3844 (1991)
  - [2] P.W. Anderson *Science* 268, 1154 (1995)
  - [3] D.G.Clarke,S.P. Strong, P.W. Anderson *Phys. Rev. Lett.* 74,4499 (1995)
  - [4] T. Giamarchi *Chem. Rev.* 104,5037 (2004)
  - [5] C. Bourbonnais and L. G. Caron, *Int. J. Mod. Phys. B* 5, 1033 (1991).
  - [6] V. M. Yakovenko, *JETP lett.* 56, 523 (1992)
  - [7] A.Nersesyan,A.Luther,F.Kusmartsev *F. Phys.Lett.A*, 176,363 (1993)
  - [8] M. Fabrizio, *Phys. Rev. B* 48, 15838 (1993).
  - [9] M. Finkelstein and A. I. Larkin, *Phys. Rev. B* 47, 10461 (1993).
  - [10] L. Balents and M. P. A. Fisher, *Phys. Rev. B* 53, 12133 (1996)
  - [11] H.J.Schulz *Phys. Rev. B* 53, 6, R2959 (1995)
  - [12] E. Arrigoni, *Phys. Rev. Lett.* 80, 790 (1998)
  - [13] U. Ledermann and K. Le Hur, *Phys. Rev. B* 61, 2497 (2000).
  - [14] U. Ledermann, *Phys. Rev. B* 63, 165110 (2001).
  - [15] P.Donohue,M.Tsuchiizu,T.Giamamrchi,Y.Suzumura *Phys. Rev. B* 63,045121 (2001).
  - [16] S. Dusuel and B. Doucot, *Phys. Rev. B* 67, 205111 (2003).
  - [17] S. Ledowski, P. Kopietz and A. Ferraz, *Phys. Rev. B* 71, 235106 (2005).
  - [18] C.Bourbonnais, L.C. Caron. *J.Phys. (France) Lett.* 45,L55 (1984)
  - [19] X.G. Wen *Phys Rev B* 42, 6623 (1990)
  - [20] Boies D., Bourbonnais, C.,Tremblay A *Phys. Rev.Lett.* 74, 968 (1995)
  - [21] S.Capponi, D.Poliblanc,F.Mila *Phys. Rev. B* 54,24,17547 (1996)
  - [22] D.G.Clarke,S.P. Strong, P.W. Anderson *Phys. Rev. Lett.* 72,3218 (1994)
  - [23] J.Moser et al. *Eur.Phys. J B* 1,39 (1998)
  - [24] A.Georges T.Giamarchi N.Sandler *Phys. Rev. B* 61, 16393 (2000)
  - [25] G.Benfatto, G.Gallavotti *Phys. Rev. B* 42, 9967 (1990)
  - [26] G.Benfatto, V.Mastropietro. *Comm. Math. Phys.* **258**, 609 (2005).
  - [27] M. A. H. Vozmediano, M. I. Katsnelson, F. Guinea, *Physics Reports* 496, 109 (2010)
  - [28] J. Gonzalez, F. Guinea and M. A. H. Vozmediano, *Nucl. Phys. B* **424**, 595 (1994).
  - [29] A.Giuliani, V.Mastropietro, M.Porta. *Annales H. Poincare'* 11, 1409 (2010)
  - [30] A.Giuliani, V.Mastropietro, M.Porta. *Phys.Rev.B* 82,

- 121418 (2010)
- [31] E. McCann, V.I. Fal'ko *Phys. Rev. Lett.* **96**, 086805 (2006)
  - [32] E. V. Castro, K. S. Novoselov, S. V. Morozov, N. M. R. Peres, J. Lopes dos Santos, J. Nilsson, F. Guinea, A. K. Geim, and A. H. C. Neto, *Phys. Rev. Lett.* **73**, 214418 (2006).
  - [33] Y. Barlas and K. Yang, *Phys. Rev. B* **80**, 161408(R) (2009).
  - [34] R. Nandkishore and L. Levitov, *Phys. Rev. B* **82**, 115431 (2010)
  - [35] V. N. Kotov, B. Uchoa, V. M. Pereira, A. H. Castro Neto and F. Guinea, arXiv:1012.3484.
  - [36] F.D.M. Haldane, *Phys.Rev.Lett.* **45**, 1358–1362 (1980)
  - [37] G. Benfatto, V. Mastropietro, *J. Stat. Phys.* **138**, 1084–1108 (2010).